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The spectrum of multiplets with two off-shell central charges

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Abstract. We analyse the spectrum of multiplets with two central charges which vanish on-shell, and show that the general spin-reducing constraint still leaves an infinite number of propagating modes of the same mass. We perform our analysis by deriving field equations from actions involving integration over the central charge dimensions. The implications of our results for bypassing the $N = 3$ barrier are briefly considered.

1. Introduction

The existence of an $N = 3$ barrier for the construction of superfield versions of N-extended supergravities (N-SGRs) or super Yang–Mills theories (N-SYMs) was proved recently (Rivelles and Taylor 1982a) under very general assumptions, and shown to persist in higher dimensions where there is even a barrier at $N = 1$ in most cases (see also Rocek and Siegel 1980, Rivelles and Taylor 1981, Taylor 1982a). Since a superfield framework is crucial for determining the ultraviolet divergence features of N-SGRs or N-SYMs the presence of the $N = 3$ barrier indicates the need to introduce new techniques beyond the traditional ones to avoid it. The detailed assumptions needed to derive the existence of the $N = 3$ barrier indicate how this might be achieved.

The basic reason for the existence of the $N = 3$ barrier is that the number of degrees of freedom of the spinor generators $\{S^i_\alpha, S_{\dot\alpha i}\}$ of the N-extended supersymmetry algebra (N-SUSY; we use chiral notation), with $1 \leq i \leq N$, increases too rapidly on N for satisfactory compensation of unphysical fields in full N-SUSY multiplets to occur. Thus besides the required physical fields of, say, N-SYM or N-SGR, there will also be other propagating fields with undesirable properties if the complete multiplets of N-SUSY are used. We may prevent this in (at least) one of three ways.

Firstly, it is possible to reduce the SUSY of the superfields used to describe the N-SYM or N-SGR to be those of $N/2$-SUSY. This approach has been used to advantage in a proof (Howe *et al* 1982) of the finiteness of $N = 4$ SYM to all orders in perturbation theory, and may be used to construct an $N = 4$ SUSY version of $N = 8$ SGR (Bufton and Taylor 1983a). However, in this case the use of supergraph Feynman rules shows (Grisaru and Siegel 1982) that the resulting theory is not expected to be finite to higher than three loops. We conclude that the loss of the full SUSY is too great to give a superfield theory which is powerful enough to handle the UV divergence problem of N-SGRs.

It is possible to penetrate the $N=3$ barrier whilst at the same time preserving the full internal symmetry of the N-SGR or N-SYM by sacrificing explicit Lorentz covariance. This destruction occurs naturally in the light-cone gauge, for which the light-cone supersymmetry subalgebra (N-LCSUSY) has spinorial generators which are only half the dimension of those of N-SUSY. In this way a full $N=4$ light-cone superfield version of $N=4$ SYM was constructed recently (Brink *et al* 1982a, Mandelstam 1982) and used to prove the UV finiteness of $N=4$ SYM to all orders of perturbation theory (Mandelstam 1982, Brink *et al* 1982b). This theory has been extended to include explicit 'soft' mass terms (preserving the UV finiteness) for spinors and scalars (Taylor 1982b, Namazie *et al* 1982, Parkes and West 1983, Rajpoot *et al* 1983) so as to give a physically more attractive and applicable theory. Similarly, the construction of N-SGR appears possible in the light cone but, as for $N/2$-SUSY methods, fails to be powerful enough to prove finiteness of the resulting superfield perturbation expansion (Taylor 1982c).

We are left with the possibility of modifying the underlying N-SUSY algebra by the addition of new generators called 'central charges', since they commute with all generators of the algebra. Thus we have that the N-SUSY algebra in the presence of central charges is defined by

$$[S^i_\alpha, S^j_\beta]_+ = 2\varepsilon_{\alpha\beta}Z^{ij} \tag{1.1}$$

$$[S_{\dot\alpha i}, S_{\dot\beta j}]_+ = 2\varepsilon_{\dot\alpha\dot\beta}\bar Z^{ij}. \tag{1.2}$$

The $\frac{1}{2}N(N-1)$ (complex) operators Z^{ij}, with $Z^{ji}=-Z^{ij}$, on the RHS of (1.1) commute with S^i_α, $S_{\dot\alpha i}$ and $P_\mu, J_{\mu\nu}$, the remaining generators of the N-SUSY algebra. They do not do so with the generators of the automorphism group SU(N) of the N-SUSY algebra without central charges, reducing this group to at most USp(N) if not to a much smaller group.

Analysis of the representations of the N-SUSY algebra with central charges shows (Sohnius 1978, Taylor 1980, Ferrara and Savoy 1982, Rands and Taylor 1983) that if the Dirac condition

$$\not{p}S^i_\alpha = Z^{ij}S_{\dot\alpha j} \tag{1.3}$$

and its complex conjugate are satisfied on a representation then only S^i_α is needed as the spinor generator of the algebra and $S_{\dot\alpha i}$ may be dispensed with. Spin reduction will therefore have occurred, and the $N=3$ barrier will have been bypassed.

The condition (1.3) and its complex conjugate leads to the requirement

$$Z^{ij}\bar Z^{jk} = p^2\delta^i_k. \tag{1.4}$$

We may represent Z^{ij} and $\bar Z^{ij}$ as differential operators on a space of functions z_{ij} and $\bar z^{ij}$, with

$$Z^{ij} = \partial/\partial z_{ij} \qquad \bar Z^{ij} = \partial/\partial \bar z^{ij} \tag{1.5}$$

so that (1.4) becomes the massless wave equation in a space–time of one time and $[3+N(N-1)]$ space dimensions. It is well known that massless irreps of N-SUSY always have half the spinorial dimensions that massive ones do, thus further explaining the spin reduction process for irreps satisfying (1.3).

In order to use the above central charges to avoid the $N=3$ barrier we have therefore to incorporate the constraints (1.3) or (1.4) on certain (not necessarily all) multiplets being used to construct the N-SYM or N-SGR. In particular, we expect to use compensating multiplets satisfying (1.3) to remove the non-local constraints on the N-SUSY multiplet containing the Yang–Mills gauge vector or the graviton for

N-SYM and N-SGR respectively, this latter multiplet itself having no central charges. Candidate compensating multiplets of this nature have already been described elsewhere by one of us (Taylor 1981, 1982d), so that such a programme does have some chance of success.

Having said that, we recognise that there is a new problem when central charges are present. We have to construct a supergeometric framework into which multiplets satisfying either $Z^{ij} \equiv 0$ or (1.4) must be naturally incorporated. If we introduce the real variables X_{ij}, Y_{ij} with $Z_{ij} = X_{ij} + \mathrm{i} Y_{ij}$, then our problem is to construct a unified framework in $[4+N(N-1)]$ dimensions (x^μ, x_{ij}, y_{ij}) for fields which either depend trivially on x_{ij}, y_{ij} or satisfy the massless wave equation in the total number of dimensions. We must have a large enough structure into which both alternatives can be embedded. We must also require that the resulting four-dimensional equations of motion are those expected for N-SYM or N-SGR.

The two criteria above, of (a) triviality or masslessness and (b) correct on-shell dimensionally reduced theory, seem mutually incompatible. Thus we might consider the theory in the total $[4+N(N-1)]$ dimensions as of Kaluza–Klein form, but the standard method of dimensional reduction by triviality of all fields (or some specified dependence on the extra dimensions), or by spontaneous compactification, is not available due to the need for keeping massless fields in the higher dimensions. Dimensional reduction by Legendre transformation (Sohnius *et al* 1981) is also unacceptable since that method only works for one extra dimension, and we would seem to require at least two. Moreover, we have to be able to encompass *both* trivial *and* massless representations during dimensional reduction, again something which cannot be achieved by dimensional reduction.

Very recently we have been able to construct a dynamical theory which uses integration over certain of the extra central charge dimensions (Restuccia and Taylor 1983). The region of integration in these variables is over a suitable cone Γ, so that the four-dimensional space–time $\boldsymbol{R}^4$ is at the vertex of the cone. Actions have been constructed by integration over the interior of Γ and shown to give suitable field equations in $\boldsymbol{R}^4$ for a wide variety of fields. In particular, superspace actions were shown to be most naturally given by integration over Γ for the bosonic variables.

One of the unanswered problems in the above discussion was the expected spectrum of representations satisfying (1.3) (or (1.4)) without any further constraints. For simplicity we will not consider the fermionic aspects of the problem, and so take a scalar field ϕ satisfying (1.4). What are the expected field equations in $\boldsymbol{R}^4$ for such a field? The answer to this question is relevant to the problem of constrained supergeometry for N-SYM or N-SGR beyond the $N=3$ barrier. We will be more precise about this relationship in § 5. Before that we present, in § 2, a construction of a suitable constrained action for one extra central-charge dimension. This is extended in § 3 to the case of two central charges with extra constraints beyond (1.4). Both of these analyses are a complementary approach to that given earlier (Restuccia and Taylor 1983), and lead to a final answer, which we give in § 4, to the problem as to the spectrum of ϕ without any extra constraints beyond (1.4).

2. One central charge

We begin our discussion with a consideration of the scalar field $\phi(x, x^5)$ which satisfies the massless wave equation

$$\Box\phi = \partial_5^2\phi. \tag{2.1}$$

In order to construct an action involving integration over $\boldsymbol{R}^4$ and x^5 but yet with field equation reducing purely to

$$\Box\phi(x,0)=\partial_5\phi(x,0)=0 \tag{2.2}$$

we use the following approach (already indicated in Restuccia and Taylor (1983)). We write down a four-dimensional action density L_4 with fields being the boundary values of ϕ required to specify a solution to (2.1), that is, $\phi(x,0)$ and $\partial_5\phi(x,0)$. We also require that L_4 give the equations of motion (2.2) when these boundary values are regarded as independent. Such a Lagrangian is

$$L_4=\phi(x,0)\Box\phi(x,0)-\tfrac{1}{2}[\partial_5\phi(x,0)]^2. \tag{2.3}$$

We now re-express the associated action $A_4=\int L_4\,\mathrm{d}^4x$ as an integral over the half-space $x^5\geqslant 0$ (or more generally $x^5\geqslant x^5_{(0)}$) by differentiating L_4, with $\phi(x,0)$ and $\partial_5\phi(x,0)$ now replaced by $\phi(x,x^5)$, $\partial_5\phi(x,x^5)$ and using the constraint (2.1). Thus we construct

$$A_5=\int \mathrm{d}^4x\int_{x^5\geqslant 0}\mathrm{d}x^5\,\partial_5[\phi\Box\phi-\tfrac{1}{2}(\partial_5\phi)^2]=\int_{x^5\geqslant 0}\mathrm{d}^5x\,\phi\,\partial_5\Box\phi \tag{2.4}$$

which agrees with the constrained action of Restuccia and Taylor (1983) in this case (where we assume all fields vanish at $x^5=\infty$). In order to reduce (2.1) to a first-order constraint we introduce the fields $P=\phi$, $A=\partial_5\phi$, and define the two-component vector u with $u^{\mathrm{T}}=(P,A)$ so that

$$L_4=u^{\mathrm{T}}Mu \tag{2.5}$$

where $M=\mathrm{diag}(\Box,-\tfrac{1}{2})$. We call M the *dynamical matrix*, since the propagation or vanishing of the fields on-shell is described purely in terms of its diagonal elements.

The second-order constraint (2.1) is now reduced, in terms of u, to the first-order constraint

$$\partial_5 u=M_5u \qquad M_5=\begin{pmatrix}0 & 1\\ \Box & 0\end{pmatrix}. \tag{2.6}$$

We term M_5 the *constraint matrix*, which is effectively the square root of the constraint (2.1), since $M_5^2=\Box$.

We may now determine the matrices $M_5^{\mathrm{T}}M$ and MM_5 as

$$M_5^{\mathrm{T}}M=\begin{pmatrix}0 & -\tfrac{1}{2}\Box\\ \Box & 0\end{pmatrix} \qquad MM_5=\begin{pmatrix}0 & \Box\\ -\tfrac{1}{2}\Box & 0\end{pmatrix}. \tag{2.7}$$

Then it is easy to see that

$$u^{\mathrm{T}}(M_5^{\mathrm{T}}M+MM_5)u=\phi\,\partial_5\Box\phi \tag{2.8}$$

so that we may rewrite the five-dimensional action in terms of the dynamical and constraint matrices as

$$A_5=\int_{x^5\geqslant 0}\mathrm{d}^5x[(M_5u)^{\mathrm{T}}Mu+u^{\mathrm{T}}M(M_5u)]. \tag{2.9}$$

The particular form of (2.9) will be of value in extending to the higher-dimensional cases, as we shall see shortly. Before we turn to that, we note that the equations of motion (2.2) may be derived directly from (2.9) with the constraint (2.6) using the techniques of constrained optimisation theory as in Restuccia and Taylor (1983). Thus the constraint (2.6) is implemented by a Lagrange multiplier λ. The resulting variational equations, taken with respect to the part v of u vanishing at $x^5=0$ (which forms a vector space) and that which is the non-vanishing boundary value u_0 at $x^5=0$, are (2.6) and

$$2(M_5^{\mathrm{T}}M+MM_5)u-(\partial_5+M_5^{\mathrm{T}})\lambda=0 \tag{2.10}$$

$$\int \mathrm{d}x^5[2(M_5^{\mathrm{T}}M+MM_5)-M_5^{\mathrm{T}}]u=0. \tag{2.11}$$

Inserting (2.10) in (2.11) shows that $\lambda(x,0)=0$, and since λ satisfies (from (2.6) and (2.10)) the second-order differential equation

$$(\partial_5 M_5)(M_5^{\mathrm{T}}M+MM_5)^{-1}(\partial_5+M_5^{\mathrm{T}})\lambda=0 \tag{2.12}$$

the only solution with vanishing boundary values at $x^5=0$ and $x^5=\infty$ will be $\lambda\equiv 0$. Thus $(M_5^{\mathrm{T}}M+MM_5)u=0$, so that taking $x^5=0$, and $u(x,0)=(\phi(x),\partial_5\phi(x))$,

$$\Box\phi(x)=0.$$

To obtain the remaining field equation $\partial_5\phi(x)=0$ in (2.1) we have to be more precise about the detailed behaviour of u_1 and u_2 at $x^5=\infty$ (Restuccia and Taylor 1983). For a fixed function $\alpha(x^5)$ with $\alpha(0)=1$, $\alpha(\infty)=0$ we take $u^{\mathrm{T}}=(v_1(x,x^5)+\phi(x), v_2(x,x^5)+\alpha(x^5)\partial_5\phi'(x))$, with $\partial_5\phi(x)=\phi'$, and $v(x,0)=v(x,\infty)=0$. The vanishing of λ is unchanged, but the constraint (2.6) now implies

$$\partial_5(v_2+\alpha\phi')=\partial_5(v_1+\phi)=0. \tag{2.13}$$

The first equation in (2.13) requires $v_2+\alpha\phi'=c(x)$, and from the conditions on v_2 and α at $x^5=\infty$ we must have $c(x)=0$. From the vanishing of $v_2(x,0)$ then $\phi'=0$, as required. The second of the equations (2.13) only provides the information $v_1=0$.

3. Two central charges and further constraints

In two extra dimensions x^5, x^6 the constraint (1.4) is

$$\partial_5^2+\partial_6^2=\Box. \tag{3.1}$$

The boundary values required to specify a solution of the elliptic equation (3.1) are expected to involve at least two arbitrary functions defined on a one-dimensional surface in the x^5–x^6 space. We wish to represent the boundary values by functions of x alone, so as to be evaluated at $x^5=x^6=0$. One way to achieve that has already been given for the case of $N=4$ SUSY (Bufton and Taylor 1983b), where it was shown to give an irreducible representation with spin reducing properties and two central charges. To achieve this reduction we need to impose the further constraints

$$\partial_5^2=a\Box \qquad \partial_6^2=(1-a)\Box \tag{3.2}$$

for some constant a. Then the boundary values required to specify a solution of (3.2) have been reduced to $\phi(x,\mathbf{0})\partial_5\phi(x,\mathbf{0})$, $\partial_6\phi(x,\mathbf{0})$ and $\partial_5\partial_6\phi(x,\mathbf{0})$. From the case of one extra dimension discussed in § 2 and the detailed spectrum of the two-central-charge $N=4$ SUSY irrep we expect the field equations in this case to describe two propagating and two auxiliary scalars. These will therefore be the field equations in $\boldsymbol{R}^4$:

$$\Box\phi(x,\mathbf{0}) = \Box[(\partial_5\partial_6/\Box)\phi(x,\mathbf{0})] = \partial_5\phi(x,\mathbf{0}) = \partial_6\phi(x,\mathbf{0}). \tag{3.3}$$

These equations arise from the four-dimensional Lagrange density

$$L_4 = \phi(x,\mathbf{0})\Box\phi(x,\mathbf{0}) + \Box^{-1}\partial_5\partial_6\phi(x,\mathbf{0})\Box\Box^{-1}\partial_5\partial_6\phi(x,\mathbf{0}) - \tfrac{1}{2}[\partial_5\phi(x,\mathbf{0})]^2 - \tfrac{1}{2}[\partial_6\phi(x,\mathbf{0})]^2. \tag{3.4}$$

As in § 2 we define a four-component vector u with $u^{\mathrm{T}} = (P_1, P_2, A_1, A_2)$ where $P_1 = \phi$, $P_2 = \Box^{-1}\partial_5\partial_6\phi$, $A_1 = \partial_5\phi$, $A_2 = \partial_6\phi$. Then the dynamical matrix M similar to the one-dimensional case (2.5) is

$$M = \mathrm{diag}(\Box, \Box, -\tfrac{1}{2}, -\tfrac{1}{2}) \qquad L_4 = u^{\mathrm{T}}Mu \tag{3.5}$$

where all fields in L_4 are evaluated at $x_5 = x_6 = 0$.

We may now reduce the constraints (3.2) to first-order form in u as

$$\partial_5 u = M_5 u \qquad \partial_6 u = M_6 u \qquad \partial_5\partial_6 u = M_{56} u \tag{3.6}$$

where the constraint matrices M_5, M_6, M_{56} are obtained as

$$M_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \\ a\Box & 0 & 0 & 0 \\ 0 & \Box & 0 & 0 \end{pmatrix} \qquad M_6 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & (1-a) & 0 \\ 0 & \Box & 0 & 0 \\ (1-a)\Box & 0 & 0 & 0 \end{pmatrix} \qquad M_{56} = M_5 M_6. \tag{3.7}$$

We now use the procedure of § 2 to construct

$$\int_{x^5\geqslant 0, x^6 \geqslant 0} \mathrm{d}^4x\, \mathrm{d}x^5\, \mathrm{d}x^6\, \partial_5\partial_6[u^{\mathrm{T}}Mu] \tag{3.8}$$

using the constraints (3.6). In terms of the components of u, (3.8) may be shown directly to be

$$[2 + a(1-a)] \int_{x^5\geqslant 0, x^6\geqslant 0} \mathrm{d}^6x \{P_2\Box^2 P_1 + A_2\Box A_1\}. \tag{3.9}$$

Direct evaluation of the relevant matrix products similar to (2.7) allows us to show that (3.9) is identical to the expression involving u only:

$$A_6 = \int_{x^5\geqslant 0, x^6\geqslant 0} \mathrm{d}^6x [(M_{56}u)^{\mathrm{T}}Mu + (M_5u)^{\mathrm{T}}M(M_6u) + (M_6u)^{\mathrm{T}}M(M_5u) + u^{\mathrm{T}}M(M_{56}u)]. \tag{3.10}$$

It is this expression, together with the constraints (3.6), which describes the theory in six dimensions. The derivation of the field equations (3.3) proceeds in a manner very similar to that for the case of one extra dimension, so we will not give it here.

4. Two central charges

We are now able to construct the appropriate Lagrangians in six dimensions which arise solely from the constraint (3.1). To do this in terms of the boundary values of $\phi(x, x^5, x^6)$ at the vertex V: $x^5 = x^6 = 0$ of the cone Γ: $x^5 \geqslant 0$, $x^6 \geqslant 0$ we must construct a sequence of values of $\phi(x, x^5, x^6)$ and its derivatives with respect to x^5 and x^6 whose values at V would be needed to specify a solution of (3.1) inside Γ. There must be an infinite number of these, since, as we remarked in § 3, two functions on a one-dimensional curve in V are expected to be required.

We consider, therefore, the sequence

$$u^{\mathrm{T}} = (\phi, \Box^{-1}\partial_5\partial_6\phi, \partial_5\phi, \partial_6\phi, \Box^{-1}\partial_5^2\phi, \Box^{-2}\partial_5^3\partial_6\phi, \Box^{-1}\partial_5^3\phi, \ldots). \quad (4.1)$$

We have used (3.1) to replace ∂_6^2 by $(\Box - \partial_5^2)$, so that we expect the values of (4.1) to be necessary at V to allow a solution of (3.1) in Γ. Moreover, if we wish to reduce (3.1) to a first-order constraint we cannot take a finite subset of (4.1); action of ∂_5, ∂_6 or $\partial_5\partial_6$ on any subset of the terms in (4.1) will always create elements outside that subset. Therefore u is the minimum specification of ϕ at V needed for first-order constraints. This latter is required so as to allow a well posed optimisation problem to be constructed.

Let us define the physical fields, for $n \geqslant 0$,

$$P_1(n) = \Box^{-n}\partial_5^{2n}\phi \qquad P_2(n) = \Box^{-(n+1)}\partial_5^{2n+1}\phi \quad (4.2)$$

and the auxiliary fields

$$A_1(n) = \Box^{-n}\partial_5^{2n+1}\phi \qquad A_2(n) = \Box^{-n}\partial_5^{2n}\partial_6\phi. \quad (4.3)$$

Then a four-dimensional action density embodying all the fields in (4.2) and (4.3) is

$$L_4 = \sum_{n \geqslant 0} [P_1(n)\Box P_1(n) + P_2(n)\Box P_2(n) - \tfrac{1}{2}A_1(n)^2 - \tfrac{1}{2}A_2(n)^2]. \quad (4.4)$$

We note that (4.4) will have as field equations at V

$$\Box P_1(n) = \Box P_2(n) = A_1(n) = A_2(n) = 0 \quad (4.5)$$

(where all the fields in (4.4) and (4.5) are evaluated at $x^5 = x^6 = 0$). In other words, we are constructing a six-dimensional version of a theory with an infinite number of propagating massless scalar fields. Such a feature appears unavoidable; all the fields in (4.2) and (4.3) are required, as we remarked above, and the only expressions involving them all with the correct dimension for an action are those of (4.4) (to within trivial normalisation factors). We are thus forced to consider such infinite multiplets.

In order to construct the six-dimensional action we note first from (4.4) that the dynamical matrix is the infinite matrix

$$M = \mathrm{diag}(\Box, \Box, -\tfrac{1}{2}, -\tfrac{1}{2}; \Box, \Box, -\tfrac{1}{2}, -\tfrac{1}{2}; \Box, \ldots). \quad (4.6)$$

The constraint matrices embodying (3.1) may be calculated directly from the formulae

$$\begin{aligned} &\partial_5 P_1(n) = A_1(n) \qquad &&\partial_5 P_2(n) = A_2(n+1) \\ &\partial_5 A_1(n) = \Box P_1(n+1) \qquad &&\partial_5 A_2(n) = \Box P_2(n) \end{aligned} \quad (4.7)$$

$$\begin{aligned} &\partial_6 P_1(n) = A_2(n) \qquad &&\partial_6 P_2(n) = A_1(n) - A_1(n+1) \\ &\partial_6 A_1(n) = \Box P_2(n) \qquad &&\partial_6 A_2(n) = \Box P_1(n) - \Box P_1(n+1) \end{aligned} \quad (4.8)$$

$$\begin{aligned} &\partial_5\partial_6 P_1(n) = \Box P_2(n) \qquad &&\partial_5\partial_6 P_2(n) = \Box P_1(n+1) - \Box P_1(n+2) \\ &\partial_5\partial_6 A_1(n) = \Box A_2(n+1) \qquad &&\partial_5\partial_6 A_2(n) = \Box A_1(n) - \Box A_1(n+1). \end{aligned} \quad (4.9)$$

We see from (4.7), (4.8) and (4.9) the justification of our earlier statement that first-order constraints require the whole of (4.1). We may now construct the constraint matrices M_5, M_6, M_{56} so that (3.1) reduces to (3.6) as the infinite matrices constructed from (4×4) blocks labelled by n in (4.7)–(4.9). The diagonal blocks of M_5 and M_6 are respectively

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \square & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \square & 0 & 0 \\ \square & 0 & 0 & 0 \end{pmatrix} \tag{4.10}$$

whilst the only other non-zero blocks in M_5 are those relating the nth row block and the $(n+1)$th column block as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \square & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.11a}$$

and relating the same blocks in M_6 by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -\square & 0 & 0 & 0 \end{pmatrix}. \tag{4.11b}$$

Finally

$$M_{56} = M_5 M_6. \tag{4.12}$$

We may now construct the six-dimensional action, as in the previous sections, as

$$\begin{aligned} \int_\Gamma \mathrm{d}^6x\ \partial_5\partial_6(u^{\mathrm{T}}Mu) \\ &= \int_\Gamma \mathrm{d}^6x \sum_{n\geq 0} [P_1(n)\square^2 P_2(n) + 2P_2(n)\square^2 P_1(n+1) - 2P_2(n)\square^2 P_1(n+2) \\ &\quad + A_1(n)\square A_2(n) + A_1(n)\square A_2(n+1) + A_2(n)\square A_1(n+1) \\ &\quad - 2A_1(n+1)\square A_2(n+1)] \end{aligned} \tag{4.13}$$

where the last line of (4.13) arises from use of the constraint (3.1). We finally may show, by direct computation, that (4.13) is identical to

$$A_6 = \int_\Gamma \mathrm{d}^6x [(M_{56}u)^{\mathrm{T}}Mu + (M_5u)^{\mathrm{T}}M(M_6u) + (M_6u)^{\mathrm{T}}M(M_5u) + u^{\mathrm{T}}M(M_{56}u)]. \tag{4.14}$$

This is itself identical to the form (3.10) for two central charges with the further constraint (3.2). The action (4.14) and the constraints (3.6), (4.10)–(4.12) are the basic constructions of the paper.

We may now proceed from the action A_6 to deduce the equations of motion (4.5) in $\boldsymbol{R}^4$, using the optimisation procedure of Restuccia and Taylor (1983) which we outlined for the one-dimensional case. Thus the variational equations are (3.6), (4.10)–(4.12) and

$$2Nu-(\partial_5+M_5^{\mathrm{T}})\lambda_5-(\partial_6+M_6^{\mathrm{T}})\lambda_6=0 \tag{4.15}$$

and

$$\int \mathrm{d}x^5\,\mathrm{d}x^6(2Nu-M_5^{\mathrm{T}}\lambda_5-M_6^{\mathrm{T}}\lambda_6)=0. \tag{4.16}$$

We have dropped the third of the constraints (3.6) since it follows from the first two. We may solve these constraints as

$$(\partial_5+M_5^{\mathrm{T}})\lambda_5+(\partial_6+M_6^{\mathrm{T}})\lambda_6=N\exp(M_5x_5+M_6x_6)\Lambda_0 \tag{4.17}$$

where Λ_0 is independent of x_5 and x_6. The boundary condition (4.16) may be rewritten, using (4.15), as

$$\int \mathrm{d}x^5\,\mathrm{d}x^6\,(\partial_5\lambda_5+\partial_6\lambda_6)=0. \tag{4.18}$$

Since λ_5 and λ_6 are assumed to vanish for large x_5 and x_6 the integral (4.18), with (4.17), can only be defined at such values of x_5 and x_6 provided $\Lambda_0=0$. Thus in (4.15)

$$Nu=0$$

so that at V the boundary values must satisfy the equations $\Box u=0$.

To obtain the vanishing of the auxiliary fields $A_i(n)$ and not just $\Box A_i(n)$ requires a more careful consideration of the limiting values of the components of u by use of the fixed function α as in § 2; since this is straightforward we will not give its details here.

Our proof has not been epsilantic in the determination of conditions that the infinite sums over n occurring in L_4 of (4.4) and in (4.13) and (4.14) converge. The use of Hilbert spaces of sequences (l_2 spaces) will naturally ensure this, though we will not give any details, which can evidently be filled in by functionally minded readers. The extensions of our results to spinors and higher spin component fields will also be clear (along the lines of discussions in Restuccia and Taylor (1983)), as will the inclusion of gauge interactions in the above scalar and spinor cases.

5. Discussions

We have shown that the spin reducing constraint (1.4), in the case of $N=2$, is not powerful enough to have only a finite number of propagating modes. The same result should be valid by almost identical methods, for the case of higher N.

We have also only shown our result for component fields, though the remark at the end of § 4 indicates its validity for all components of a supersymmetric multiplet. Indeed, this is obvious: if one component field of a multiplet has such a spectrum then the remainder must have the same spectrum. It is also possible to consider this problem in a purely supersymmetric manner in terms of the $N=2$ superfield Φ_i satisfying (Sohnius 1978)

$$D_{\alpha(i}\Phi_{j)}=D_{\dot\alpha(i}\Phi_{j)}=0. \tag{5.1}$$

The associated unconstrained full-superspace Lagrangian is

$$\int d^6x\, d^8\theta(\Phi_i^+\Phi_i + K^{\alpha ij}D_{\alpha(i}\Phi_{j)} + K^{\dot\alpha ij}D_{\dot\alpha(i}\Phi_{j)} + \text{HC}). \tag{5.2}$$

The case of (5.1), (5.2) with the additional constraint $\partial_6\Phi_i = 0$ was considered earlier (Restuccia and Taylor 1983). Without this extra constraint we will still obtain the field equations

$$\partial_5\Phi_i = \partial_6\Phi_i = 0. \tag{5.3}$$

However, there will still be the infinite set of physical fields $P_1(n)$, $P_2(n)$ of § 4, so we still have the infinite spectrum discovered in § 4.

We now turn to discuss the relevance of our result for the problem of constructing N-SYM and N-SGR for $N \geq 3$. At the linearised level we know that multiplets carrying central charges in a spin-reducing manner must be used as compensators, through field redefinition rules (Rivelles and Taylor 1982b, de Wit 1982) to remove the non-local constraints on vectors and higher spin fields. These latter absorb the centrally charged component fields either to become gauge fields of a local symmetry (gauge, translation or SUSY) or to become auxiliary, and so algebraically solvable by their equations of motion.

The question which we have now to answer is: what are torsion constraints which lead to suitable spin degenerate central charge multiplets? The result of this paper shows conclusively that these constraints cannot reduce, at the linearised level, solely to (1.3) or (1.4). There must be a loss of rotation invariance in the internal symmetry space leading to constraints of the form (3.2) or to independence in all but one of the central charge dimensions, so reducing to (2.1). For otherwise there would occur multiplets with infinite numbers of components, which would be very difficult to compensate amongst each other so as to produce a finite number of components to compensate the non-central charge multiplets. Such constraints as (3.2) or (2.1) are therefore related either to breaking the residual USp(N) invariance associated with one central charge to at most USp(N/2) due to the appearance of constants like a in (3.2), or to reducing the theory to that with only one central charge by triviality in all the others. This gives a further guiding principle in searching for suitable constraints for $N = 4$ SYM or $N \geq 3$ SYM.

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